1. Suppose that

$$\triangle f = -div(grad(f))$$

for smooth functions on a Riemannian manifold.

(a) Define the quotient

$$Q(f) = \frac{\|\nabla f\|^2}{\|f\|^2}$$

on nonzero smooth functions and show that it is defined for all nonzero $f \in H^1(M)$. This amounts to showing that if $f_n \to f \neq 0$ in the $H^1(M)$ norm, then $Q(f_n)$ is Cauchy.

(b) Use this to show that if

$$\dot{H} = \{f \in H^1(M) : \int_M f = 0\}$$

then

$$C \|\nabla f\|^2 \geq \|f\|^2, f \in \dot{H}$$

for some positive constant C.(Hint: what is the kernel of $\triangle$)

(c) Show that if $g \in L^2(M)$ with $\int_M g = 0$ then there is some $f \in \dot{H}$ solving

$$\triangle f = g$$

weakly.

(d) (Hard) Deduce that $f \in H^2(M)$. (Hint:Once you have existence in $H^1 = W^{1,2}$ use a partition of unity to work in local coordinates for regularity. Is $\triangle$ elliptic, what is its symbol?)

(e) (Hard) Compute the second variation

$$\frac{d^2}{dt^2}|_{t=0} Q(f + th)$$

where $f, h \in \dot{H}$. Use this to show that any g has a UNIQUE (weak) solution $f \in \dot{H} \cap H^2(M)$. Show that this is not unique in $H^2(M)$. There is a strict condition that becomes non-strict when considering $H^2(M)$, what is it?

(f) (Hard) Show that the operator

$$\triangle + 1 : H^2(M) \to L^2(M)$$

is invertible. (Note: You can therefore, using Rellich-Kondrachov and the theory of compact operators, deduce that there is an orthonormal basis of eigenvectors as in the next problem.)

2. Suppose that there exists an orthonormal basis $\{v_i\}$ of eigenvectors for

$$\triangle_k : W^{1,2}\Omega^k(M) \subseteq L^2\Omega^k(M) \to L^2\Omega^k(M)$$

with corresponding eigenvalues $\{\lambda_i\}$. Then $0 \le \lambda_i \le \lambda_{i+1}$ and the only accumulation point of the eigenvectors is infinity.

With this in mind, define the heat operator on $L^2\Omega^k(M)$ by

$$e^{-t\triangle_k} f = \sum_{i=1}^{\infty} e^{-t\lambda_i} \langle v_i, f \rangle v_i$$

(a) Show this defines an element of $L^2\Omega^k(M)$, and that $e^{-t\triangle_k}$ is linear.(Hint: Check Folland's chapter on Hilbert spaces)

(b) Show that $f = f_H + f_\perp$ where $f_H \in \ker \triangle_k$ and $f_\perp \perp \ker \triangle_k$. Deduce that

$$\left\| e^{-t\triangle_k} f_\perp \right\|_{L^2} \le e^{-t\lambda_i} \left\| f_\perp \right\|_{L^2}$$

$$e^{-t\triangle_k} f_H = f_H$$

where λ_i is the lowest non-zero eigenvalue. (Hint: Any finite dimensional subspace of a Hilbert space is closed.)

(c) Conclude that $e^{-t\triangle_k} f \to f_H$ in $L^2\Omega^k(M)$

(d) (Much harder) Show that

$$\left\| e^{-t\triangle_k} f_\perp \right\|_{H^1} \leq C \left\| f_\perp \right\|_{H^1}, t > 1$$

where C depends only on the eigenvalues.

(e) (Extremely hard) Show that $e^{-t\triangle_k} f \to f_H$ in $W^{l,2}\Omega^k(M)$ for each l and deduce that if $f \in \Omega^k(M)$ is closed then f, f_H represent the same cohomology class.(You will need some sophisticated tools from Lee's chapter on de Rham cohomology, combined with the knowledge that if f is smooth then the form $\omega(x,t) = e^{-t\triangle_k} f(x)$ is smooth on the product $M \times [0, \infty)$)

3. Define L_ω as in lecture 6.

(a) Show that if $S \subseteq L^2\Omega^k(M)$ is a subspace, then its completion, defined as the set of limits of Cauchy sequences in the L^2-norm is a well-defined subspace of $L^2\Omega^k(M)$, and is a Hilbert space.

(b) (Very Hard) Show that there exists a positive constant C such that if $d\alpha = 0$ and $\alpha \perp \ker d^*$ then

$$\|\alpha\|_{L^2} \leq C \|d^*\alpha\|_{L^2}$$

and deduce that if $\beta \in L_\omega$ then

$$|\langle \beta - \omega, \alpha \rangle_{L^2}| \leq C \|\beta - \omega\|_{L^2} \|d^*\alpha\|_{L^2}$$

for any $\alpha \in \Omega^k(M)$.(Hint: For the second part show that both sides only depend on $\alpha - \alpha_{\ker d^*}$, where $\alpha_{\ker d^*}$ is the orthogonal projection.)

(c) Show that if $\beta_n \rightharpoonup \gamma$(weak convergence) where $\beta_n \in L_\omega$, then there exists a positive constant K such that

$$|\langle \gamma - \omega, \alpha \rangle_{L^2}| \leq K \|d^* \alpha\|_{L^2}$$

(d) Deduce using parts a) and c) that $\gamma \in L_\omega$